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Behavior Patterns of Observables in Quantum-Classical Limit

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It is shown that in the quantum-classical limit (QCL) the observables have different behavior patterns connected with the preservation or loss of their role of significant variables and of their stochasticity. *This result rectifies the traditional belief of a unique behavior pattern associated with the disappearance of "uncertainties."*

1. INTRODUCTION

The quantum → classical limit (QCL) is associated (Landau and Lifschitz, 1980; Messiah, 1969) *with the condition* $\hbar \to 0$ *for the Planck constant* $\hbar$. Corresponding to the QCL the observables (i.e., the variables characterizing the physical systems) are changed from a quantum to a classical form. It is largely agreed that in the mentioned forms the observables are endowed, respectively unendowed, with "uncertainties" (specific for their measurements). So one believes that in the QCL the observables have a unique behavior pattern (way) connected with the fact that their "uncertainties" become null when $\hbar \to 0$.

But by a minute reexamination of the issue one finds (Dumitru, 1977, 1987, 1988, 1991, 1993) that in fact the mentioned "uncertainties" are nothing but fluctuation parameters (FP) of observables regarded as stochastic (random) variables. Such FP are specific (Dumitru, 1977, 1987, 1988, 1991, 1993) for observables in both quantum and classical (nonquantum) contexts.

Taking into account the alluded findings, in this short paper we wish to show that *the QCL is more complex than the condition* $\hbar \to 0$ *and that in the respective limits the observables have different behavior patterns.*

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2. FLUCTUATION PARAMETERS BUT NOT "UNCERTAINTIES"

In order to follow our program let us present the traditional concept of "uncertainties" (Landau and Lifschitz, 1980; Messiah, 1969) in a more general framework of quantum mechanics. We consider a quantum system (microparticle) in a state described by the wave function ψ. Its observables are described by the operators $\hat{A}_j$ ($j = 1, 2, \ldots, n$) regarded as generalized random variables. The mean (or expected) value of $\hat{A}_j$ is $\langle \hat{A}_j \rangle_\psi = (\psi, \hat{A}_j \psi)$ with (ψ_1, ψ_2) as scalar product of ψ_1 and ψ_2. Note that ψ and $\hat{A}_j$ refer to both orbital and spin characteristics of quantum systems. Then we can define the *correlations* $\mathbb{C}_\psi(A_j, A_k)$, *dispersions* $\mathbb{D}_\psi A_j$, and *standard deviations* $\Delta_\psi A$ through the relations

$$\begin{aligned} \mathbb{C}_\psi(A_j, A_k) &= (\delta_\psi \hat{A}_j \psi, \delta_\psi \hat{A}_k \psi) \\ \delta_\psi \hat{A}_j &= \hat{A}_j - \langle \hat{A}_j \rangle_\psi \\ \mathbb{D}_\psi A_j &= \mathbb{C}(A_j, A_j) \\ \Delta_\psi A_j &= (\mathbb{D}_\psi A_j)^{1/2} \end{aligned} \tag{1}$$

Traditionally (Landau and Lifschitz, 1980; Messiah, 1969) in the case of an orbital observable A_j the quantity $\Delta_\psi A_j$ from (1) is regarded as "uncertainty." Such a view is connected with the fact that for $A_1 = A$ and $A_2 = B$ the quantities (1) satisfy the relation

$$\Delta_\psi A \cdot \Delta_\psi B \geq \left| (\delta_\psi \hat{A}\psi, \delta_\psi \hat{B}\psi) \right| \tag{2}$$

which in the case when A and B are canonically conjugated gives the famous Heisenberg *uncertainty relation*

$$\Delta_\psi A \cdot \Delta_\psi B \geq \frac{\hbar}{2} \tag{3}$$

Dumitru (1977, 1987, 1988, 1991, 1993) has proved that in fact *the quantities $\Delta_\psi A_j$ are not "uncertainties," but simple fluctuation parameters (FP) and that such FP appear both in quantum and classical (nonquantum) physics.* The mentioned proof provides (Dumitru, 1977, 1987, 1988, 1991, 1993) a natural and real support for avoiding the shortcomings which appear in the traditional conception founded on the idea of "quantum uncertainties." *Now we wish to investigate the behavior patterns of the quantities ΔA_j in the QCL.*

3. BEHAVIOR PATTERNS IN QCL

For quantum observables all the FP defined in (1) are (Dumitru, 1993) directly dependent on the Planck constant $\hbar$. So $\hbar$ plays the role of generic indicator for quantum stochasticity. On the other hand, the condition $\hbar \to 0$ is associated (Landau and Lifschitz, 1980; Messiah, 1969) with the QCL. But the QCL is somehow more complex than the condition $\hbar \to 0$. That is why we consider it to be of interest to investigate the behavior pattern of observables in the QCL.

Let us refer first to the spin observables. We consider an electron whose spin characteristics are described by the spin wave function (spinor) ψ_s given by

$$\psi_s = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \qquad \alpha \in \left[0, \frac{\pi}{2}\right] \tag{4}$$

As specific observable we take the z component of the spin angular moment $S_z = (\hbar/2)\sigma_z$ (σ_z being the corresponding Pauli matrix). Then we find

$$\Delta S_s = \frac{\hbar}{2} \sin 2\alpha \tag{5}$$

We regard ΔS_z as a FP which describes quantitatively the stochastic characteristics of the observable S_z. Then from (5) one sees that the spin quantum stochasticity is in direct dependence of $\hbar$, it being significant or not as $\hbar \neq 0$ or $\hbar \to 0$. This means that $\hbar$ plays the role of generic indicator of the mentioned kind of stochasticity. But for the state described by (4) one obtains also $\langle S_z \rangle = (\hbar/2)\cos 2\alpha$. This additional result shows that in fact when $\hbar \to 0$ the observable S_z disappears completely. On the other hand, for the discussed variable the condition $\hbar \to 0$ is identical with the QCL. Then we conclude that *for spin observables the behavior pattern in QCL consists in an annulment of both stochastic characteristics and mean values (i.e., in a complete disappearance).*

Now let us discuss the cases of orbital quantum variables. In such cases the QCL implies not only the condition $\hbar \to 0$, but also the requirement that some quantum numbers grow unboundedly. The mentioned requirement is due to the fact that some significant observables connected with the orbital motion (e.g., the energy) pass from their quantum values to adequate classical values.

As a first example of orbital observables we quote the coordinate x of a harmonic oscillator with

$$\Delta_\psi x = \left[\frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right)\right]^{1/2} \tag{6}$$

where m is the mass, ω is the angular frequency, and n is the oscillation quantum number. In such a case the QCL means not only $\hbar \to 0$, but also $n \to \infty$, so that the quantum energy $E_n = \hbar\omega(n + 1/2)$ passes to the classical energy $E = \frac{1}{2}m\omega^2 x_M^2$, where x_M is the coordinate amplitude. Then in the QCL the quantity from (6) passes to the classical value:

$$\Delta_{cl}x = \frac{x_M}{\sqrt{2}} \tag{7}$$

But $\Delta_\psi x$ and $\Delta_{cl}x$ are FP which describe the stochastic characteristic of x in quantum, respectively classical, contexts. Then it results that *in the QCL the behavior pattern of x consists in preservation of both its role of significant observable and its stochasticity.*

For another example of orbital observable we consider the distance r between the electron and nucleus in a hydrogen atom. If the electron is in a state described by the orbital wave function ψ_{nlm} with $l = n - 1$ (where n, l, and m are, respectively, the principal, orbital, and magnetic quantum numbers) for $\Delta_\psi r = \Delta r$ we can use the expression given in Schwabl (1992) rewritten in the form

$$\Delta r = \frac{2\pi\epsilon_0}{m_0 e}\hbar^2 n(n + 1)^{1/2} \tag{8}$$

with m_0 the mass and e the charge of the electron. In the mentioned situation the energy of the electron is

$$E_n = -\frac{m_0 e^4}{32\pi^2\epsilon_0^2\hbar^2 n^2} \tag{9}$$

and in the QCL it must take its classical value E_{cl}. Then in the QCL we must have $\hbar \to 0$ and $n \to \infty$ so that

$$\hbar n \to \left(\frac{-m_0 e^4}{32\pi^2\epsilon_0^2 E_{cl}}\right)^{1/2} \tag{10}$$

(note that $E_{cl} < 0$ because the electron is in a bound state). Then it results that in the QCL we have

$$\Delta r \to \left(\frac{\hbar e^4}{16\pi\epsilon_0}\right)^{1/2}(-2m_0 E_{cl})^{-1/4} \tag{11}$$

In the same circumstances we have

$$r \to r_{cl} = -\frac{e^2}{8\pi\epsilon_0 E_{cl}} \tag{12}$$

From (11) and (12) it results that in the QCL, when $\hbar \to 0$, one finds $\Delta r \to 0$ and $r \to r_{cl} \neq 0$. This means that in the QCL the behavior pattern of r

consists in a preservation of role of significant observable accompanied with a loss of stochasticity.

Now we end our considerations with the following concluding remark: in QCL the quantum observables have the following different behavior patterns:

(i) The complete disappearance of both stochastic characteristics and of mean value, as in the case of spin observables.
(ii) The preservation both of the role of significant variable and of stochastic characteristics, as in the case of oscillator coordinate x.
(iii) The preservation of the role of significant variable, but the loss of stochastic characteristics, as in the case of electron distance r.

These remarks rectify the traditional belief of a unique behavior pattern associated with the disappearance of "uncertainties" in QCL.

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